AUTOMORPHISM GROUPS OF COMPACT COMPLEX SUPERMANIFOLDS

HANNAH BERGNER AND MATTHIAS KALUS

ABSTRACT. Let $\mathcal{M}$ be a compact complex supermanifold. We prove that the set $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms of $\mathcal{M}$ can be endowed with the structure of a complex Lie group acting holomorphically on $\mathcal{M}$, so that its Lie algebra is isomorphic to the Lie algebra of even holomorphic super vector fields on $\mathcal{M}$. Moreover, we prove the existence of a complex Lie supergroup $\mathrm{Aut}(\mathcal{M})$ acting holomorphically on $\mathcal{M}$ and satisfying a universal property. Its underlying Lie group is $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and its Lie superalgebra is the Lie superalgebra of holomorphic super vector fields on $\mathcal{M}$. This generalizes the classical theorem by Bochner and Montgomery that the automorphism group of a compact complex manifold is a complex Lie group. Some examples of automorphism groups of complex supermanifolds over $\mathbb{P}_1(\mathbb{C})$ are provided.

Keywords: compact complex supermanifold; automorphism group

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1. INTRODUCTION

The automorphism group of a compact complex manifold M carries the structure of a complex Lie group which acts holomorphically on M and whose Lie algebra consists of the holomorphic vector fields on M (see [BM47]). In this article, we investigate how this result can be extended to the category of compact complex supermanifolds.

Let $\mathcal{M}$ be a compact complex supermanifold, i.e. a complex supermanifold whose underlying manifold is compact. An automorphism of $\mathcal{M}$ is a biholomorphic morphism $\mathcal{M} \to \mathcal{M}$. A first candidate for the automorphism group of such a supermanifold is be the set of automorphisms, which we denote by $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. However, every automorphism φ of a supermanifold $\mathcal{M}$ (with structure sheaf $\mathcal{O}_{\mathcal{M}}$) is "even" in the sense that its pullback $\varphi^* : \mathcal{O}_{\mathcal{M}} \to \tilde{\varphi}_*(\mathcal{O}_{\mathcal{M}})$ is a parity-preserving morphism. Therefore, we can (at most) expect this set of automorphisms of $\mathcal{M}$ to carry the structure of a classical Lie group if we require its action on $\mathcal{M}$ to be smooth or holomorphic. This way we will not receive a Lie supergroup of positive odd dimension. We will prove that the topological group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, endowed with an analogue of the compact-open topology, carries the structure of a complex Lie group such that the action on $\mathcal{M}$ is holomorphic and its Lie algebra is the Lie algebra of even holomorphic super vector fields on $\mathcal{M}$. It should be noted that the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is in general different from the group $\mathrm{Aut}(M)$ of automorphisms of the underlying manifold M. There is a group homomorphism $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(M)$ given by assigning the underlying map to an automorphism of the supermanifold; this group homomorphism is in general neither injective nor surjective.

We will find the automorphism group of a compact complex supermanifold $\mathcal{M}$ to be a complex Lie supergroup which acts holomorphically on $\mathcal{M}$ and satisfies a universal property. In analogy to the classical case, its Lie superalgebra is the Lie superalgebra of holomorphic super vector fields on $\mathcal{M}$, and the underlying Lie group is $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, the group of automorphisms

of $\mathcal{M}$. Using the equivalence of complex Harish-Chandra pairs and complex Lie supergroups (see [Vis11]), we construct the appropriate automorphism Lie supergroup of $\mathcal{M}$.

More precisely, the outline of this article is the following: First, we introduce a topology on the set $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms on a compact complex supermanifold $\mathcal{M}$ (c.f. Section §3). This topology is analogue of the compact-open topology in the classical case, which coincides in the case of a compact complex manifold with the topology of uniform convergence. We prove that the topological space $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ with composition and inversion of automorphisms as group operations is a locally compact topological group which satisfies the second axiom of countability.

In Section §4, the non-existence of small subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is proven, which means that there exists a neighbourhood of the identity in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ with the property that this neighbourhood does not contain any non-trivial subgroup. A result on the existence of Lie group structures on locally compact topological groups without small subgroups (see [Yam53]) then implies that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of real Lie group.

Then, continuous one-parameter subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and their action on the supermanifold $\mathcal{M}$ are studied (see Section §5). This is done in order to obtain results on the regularity of the $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$-action on $\mathcal{M}$ and characterize the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. We prove that the action of each continuous one-parameter subgroup of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is analytic. As a corollary we get that the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is isomorphic to the Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even holomorphic super vector fields on $\mathcal{M}$, and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group so that its natural action on $\mathcal{M}$ is holomorphic.
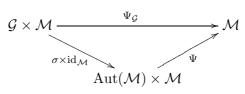
Next, we show that the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of holomorphic super vector fields on a compact complex supermanifold $\mathcal{M}$ is finite-dimensional (see Section §6). Since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group, we already know that $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$, the even part of $\mathrm{Vec}(\mathcal{M})$, is finite-dimensional. The key point in the proof in the case of a split supermanifold $\mathcal{M}$ is that the tangent sheaf of $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_M$-modules on the compact complex manifold M with $\mathcal{O}_M$ as the sheaf of holomorphic functions on M.

Let α denote the action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ by conjugation: $\alpha(\varphi)(X) = \varphi_*(X) = (\varphi^{-1})^* \circ X \circ \varphi^*$ for $\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $X \in \mathrm{Vec}(\mathcal{M})$. The restriction of this representation α to $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$, the even part of the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$, coincides with the adjoint action of the Lie group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on its Lie algebra, which is isomorphic to $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$. Hence α defines a Harish-Chandra pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$. The equivalence between Harish-Chandra pairs and complex Lie supergroups allows us to define the automorphism Lie supergroup of a compact complex supermanifold as follows (see Definition 7.1):

Definition (Automorphism Lie supergroup)**.** Define the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a compact complex supermanifold to be the unique complex Lie supergroup associated to the Harish-Chandra pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$ with representation α.

The natural action of the automorphism Lie supergroup $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$ is holomorphic, i.e. we have a morphism $\Psi : \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ of complex supermanifolds. The automorphism Lie supergroup $\mathrm{Aut}(\mathcal{M})$ satisfies the following universal property (see Theorem 7.2):

Theorem. *If $\mathcal{G}$ is a complex Lie supergroup with a holomorphic action $\Psi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$, then there is a unique morphism $\sigma : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups such that the diagram*

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{M} & \xrightarrow{\ \Psi_{\mathcal{G}}\ } & \mathcal{M} \\
{\scriptstyle \sigma \times \mathrm{id}_{\mathcal{M}}} \searrow & & \nearrow {\scriptstyle \Psi} \\
& \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} &
\end{array}
$$

is commutative.

The automorphism Lie supergroup of a compact complex supermanifold is the unique complex Lie supergroup satisfying the preceding universal property.

In the classical case, another class of complex manifolds where the automorphism group carries the structure of a Lie group is given by the bounded domains in $\mathbb{C}^m$ (see [Car79]). An analogue statement is false in the case of supermanifolds. In Section §8, we give an example showing that in the case of a complex supermanifold $\mathcal{M}$ whose underlying manifold is a bounded domain in $\mathbb{C}^m$ there does in general not exist a Lie supergroup acting on $\mathcal{M}$ and satisfying the universal property of the preceding theorem.

In Section §9, the automorphism group $\mathrm{Aut}(\mathcal{M})$ or its underlying Lie group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ are calculated for some supermanifolds $\mathcal{M}$ with underlying manifold $M = \mathbb{P}_1\mathbb{C}$.

2. Preliminaries and Notation

Throughout, we work with the "Berezin-Leĭtes-Kostant-approach" to supermanifolds (c.f. [Ber87], [Leĭ80], and [Kos77]). If a supermanifold is denoted by a calligraphic letter $\mathcal{M}$, then we denote the underlying manifold by the corresponding uppercase standard letter M, and the structure sheaf by $\mathcal{O}_{\mathcal{M}}$. We call a supermanifold $\mathcal{M}$ compact if its underlying manifold M is compact. By a complex supermanifold we mean a supermanifold $\mathcal{M}$ with structure sheaf $\mathcal{O}_{\mathcal{M}}$ which is locally, on small enough open subsets $U \subset M$, isomorphic to $\mathcal{O}_U \otimes \bigwedge \mathbb{C}^n$, where $\mathcal{O}_U$ denotes the sheaf of holomorphic functions on U. For a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ between supermanifolds $\mathcal{M}$ and $\mathcal{N}$, the underlying map $M \to N$ is denoted by $\tilde{\varphi}$ and its pullback by $\varphi^* : \mathcal{O}_{\mathcal{N}} \to \tilde{\varphi}_* \mathcal{O}_{\mathcal{M}}$. An automorphism of a complex supermanifold $\mathcal{M}$ is a biholomorphic morphism $\mathcal{M} \to \mathcal{M}$.

For a complex supermanifold $\mathcal{M}$, let $\mathcal{T}_{\mathcal{M}}$ denote the tangent sheaf of $\mathcal{M}$. The Lie superalgebra of holomorphic vector fields on $\mathcal{M}$ is $\mathrm{Vec}(\mathcal{M}) = \mathcal{T}_{\mathcal{M}}(M)$, it consists of the subspace $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even and the subspace $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$ of odd super vector fields on $\mathcal{M}$.

Let $\mathcal{M}$ be a complex supermanifold of dimension $(m|n)$, and let $\mathcal{I}_{\mathcal{M}}$ be the subsheaf of ideals generated by the odd elements in the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a supermanifold $\mathcal{M}$. As described in [Oni98], we have the filtration

$$\mathcal{O}_{\mathcal{M}} = (\mathcal{I}_{\mathcal{M}})^0 \supset (\mathcal{I}_{\mathcal{M}})^1 \supset (\mathcal{I}_{\mathcal{M}})^2 \supset \ldots \supset (\mathcal{I}_{\mathcal{M}})^{n+1} = 0$$

of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ by the powers of $\mathcal{I}_{\mathcal{M}}$. Define the quotient sheaves $\mathrm{gr}_k(\mathcal{O}_{\mathcal{M}}) = (\mathcal{I}_{\mathcal{M}})^k/(\mathcal{I}_{\mathcal{M}})^{k+1}$. This gives rise to the $\mathbb{Z}$-graded sheaf $\mathrm{gr}\,\mathcal{O}_{\mathcal{M}} = \bigoplus_k \mathrm{gr}_k(\mathcal{O}_{\mathcal{M}})$. Further $\mathrm{gr}\,\mathcal{M} = (M, \mathrm{gr}\,\mathcal{O}_{\mathcal{M}})$ is a split complex supermanifold of the same dimension as $\mathcal{M}$.

Note that $E := \mathrm{gr}_1(\mathcal{O}_{\mathcal{M}})$ defines a vector bundle on M. An automorphism φ of $\mathcal{M}$ yields a pullback φ^* on $\mathcal{O}_{\mathcal{M}}$. Following [Gre82], its reduction to the $\mathcal{O}_M$-module E yields a morphism of vector bundles $\varphi_0 \in Aut(E)$ over the reduction $\tilde{\varphi} \in Aut(M)$. Note that by [Mor58], $Aut(E)$ is a complex Lie group. On local coordinate domains U, V with $\varphi(U) \subset V$ we can identify $\mathcal{O}_{\mathcal{M}}|_V \cong \Gamma_{\Lambda E}|_V$ and $\mathcal{O}_{\mathcal{M}}|_U \cong \Gamma_{\Lambda E}|_U$ and following [Rot85] decompose $\varphi^* = \varphi_0^* \exp(Y)$ with $\mathbb{Z}$-degree preserving automorphism $\varphi_0^* : \Gamma_{\Lambda E}|_V \to \Gamma_{\Lambda E}|_U$ induced by φ_0. Here Y is an even superderivation on $\Gamma_{\Lambda E}|_V$ increasing the $\mathbb{Z}$-degree by 2 or more. Note that the exponential series is finite since Y is nilpotent.

3. The topology on the group of automorphisms

Let $\mathcal{M}$ be a compact complex supermanifold. An automorphism of $\mathcal{M}$ is a biholomorphic morphism $\varphi : \mathcal{M} \to \mathcal{M}$. Denote by $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ the set of automorphisms of $\mathcal{M}$.

In this section, a topology on $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is introduced, which generalizes the compact-open topology and topology of compact convergence of the classical case. Then we show that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a locally compact topological group with respect to this topology.

Let $K \subseteq M$ be a compact subset such that there are local odd coordinates $\theta_1, \ldots, \theta_n$ for $\mathcal{M}$ on an open neighbourhood of K. Moreover, let $U \subseteq M$ be open and $f \in \mathcal{O}_{\mathcal{M}}(U)$, and let U_ν be open subsets of $\mathbb{C}$ for $\nu \in (\mathbb{Z}_2)^n$. Let $\varphi : \mathcal{M} \to \mathcal{M}$ be an automorphism with $\tilde{\varphi}(K) \subseteq U$. Then there are holomorphic functions $\varphi_{f,\nu}$ on a neighbourhood of K such that

$$\varphi^*(f) = \sum_{\nu \in (\mathbb{Z}_2)^n} \varphi_{f,\nu} \theta^\nu.$$

Let

$$\Delta(K, U, f, \theta_j, U_\nu) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \,|\, \tilde{\varphi}(K) \subseteq U, \, \varphi_{f,\nu}(K) \subseteq U_\nu\},$$

and endow $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ with the topology generated by sets of this form, i.e. the sets of the form $\Delta(K, U, f, \theta_j, U_\nu)$ form a subbase of the topology.

Remark 3.1. In particular, the subsets of the form $\Delta(K, U) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \,|\, \tilde{\varphi}(K) \subseteq U\}$ are open for $K \subseteq M$ compact and $U \subseteq M$ open. Hence the map $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(M)$, associating to an automorphism φ of $\mathcal{M}$ the underlying automorphism $\tilde{\varphi}$ of M, is continuous.

Remark 3.2. The group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ endowed with the above topology is a second-countable Hausdorff space since M is second-countable.

Let $U \subseteq M$ be open. Then we can define a topology on $\mathcal{O}_{\mathcal{M}}(U)$ as follows: If $K \subseteq U$ is compact such that there exist odd coordinates $\theta_1, \ldots, \theta_n$ on a neighbourhood of K, write $f \in \mathcal{O}_{\mathcal{M}}(U)$ on K as $f = \sum_\nu f_\nu \theta^\nu$. Let $U_\nu \subseteq \mathbb{C}$ be open subsets. Then define a topology on $\mathcal{O}_{\mathcal{M}}(U)$ by requiring that the sets of the form $\{f \in \mathcal{O}_{\mathcal{M}}(U) \,|\, f_\nu(K) \subseteq U_\nu\}$ are a subbase of the topology. A sequence of functions f_k converges to f if and only if in all local coordinate domains with odd coordinates $\theta_1, \ldots, \theta_n$ and $f_k = \sum_\nu f_{k,\nu} \theta^\nu$, $f = \sum_\nu f_\nu \theta^\nu$, the coefficient functions $f_{k,\nu}$ converge uniformly to f_ν on compact subsets. Note that for any open subsets $U_1, U_2 \subseteq M$ with $U_1 \subset U_2$ the restriction map $\mathcal{O}_{\mathcal{M}}(U_2) \to \mathcal{O}_{\mathcal{M}}(U_1)$, $f \mapsto f|_{U_1}$, is continuous. Using Taylor expansion (in local coordinates) of automorphisms of $\mathcal{M}$ we can deduce the following lemma:

Lemma 3.3. *A sequence of automorphisms $\varphi_k : \mathcal{M} \to \mathcal{M}$ converges to an automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ with respect to the topology of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ if and only if the following condition is satisfied: For all $U, V \subseteq M$ open subsets of M such that V contains the closure of $\tilde{\varphi}(U)$, there is an $N \in \mathbb{N}$ such that $\tilde{\varphi}_k(U) \subseteq V$ for all $k \geq N$. Furthermore, for any $f \in \mathcal{O}_{\mathcal{M}}(V)$ the sequence $(\varphi_k)^*(f)$ converges to $\varphi^*(f)$ on U in the topology of $\mathcal{O}_{\mathcal{M}}(U)$.*

Lemma 3.4. *If $U, V \subseteq M$ are open subsets, $K \subseteq M$ is compact with $V \subseteq K$, then the map*

$$\Delta(K, U) \times \mathcal{O}_{\mathcal{M}}(U) \to \mathcal{O}_{\mathcal{M}}(V), (\varphi, f) \mapsto \varphi^*(f)$$

is continuous.

Proof. Let $\varphi_k \in \Delta(K, U)$ be a sequence of automorphisms of $\mathcal{M}$ converging to $\varphi \in \Delta(K, U)$, and $f_l \in \mathcal{O}_{\mathcal{M}}(U)$ a sequence converging to $f \in \mathcal{O}_{\mathcal{M}}(U)$. Choosing appropriate local coordinates and using Taylor expansion of the pullbacks $(\varphi_k)^*(f_l)$, it can be shown that $(\varphi_k)^*(f_l)$ converges to $\varphi^*(f)$ as $k, l \to \infty$. This uses that the derivatives of a sequence of uniformly converging holomorphic functions also uniformly converge. $\square$

Lemma 3.5. *The topological space $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is locally compact.*

Proof. Let $\psi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$. For each fixed $x \in M$ there are open neighbourhoods V_x and U_x of x and $\tilde{\psi}(x)$ respectively such that $\tilde{\psi}(K_x) \subseteq U_x$ for $K_x := \overline{V}_x$. We may additionally assume that there are local odd coordinates $\xi_1, \ldots, \xi_n$ for $\mathcal{M}$ on U_x, and $\theta_1, \ldots, \theta_n$ local odd coordinates on an open neighbourhood of K_x. For any automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ with $\tilde{\varphi}(K_x) \subseteq U_x$, let $\varphi_{j,k}, \varphi_{j,\nu}$ (for $||\nu|| = ||(\nu_1, \ldots, \nu_n)|| = \nu_1 + \ldots + \nu_n \geq 3$) be local holomorphic functions such that

$$\varphi^*(\xi_j) = \sum_{k=1}^{n} \varphi_{j,k}\theta_k + \sum_{||\nu|| \geq 3} \varphi_{j,\nu}\theta^\nu.$$

Choose bounded open subsets $U_{j,k}, U_{j,\nu} \subset \mathbb{C}$, such that $\psi_{j,k}(x) \in U_{j,k}$ and $\psi_{j,\nu}(x) \in U_{j,\nu}$. Since ψ is an automorphism, we have

$$\det\left((\psi_{j,k}(y))_{1 \leq j,k \leq n}\right) \neq 0$$

for all $y \in K_x$. For later considerations shrink $U_{j,k}$ such that $\det(C) \neq 0$ for all $C = (c_{j,k})_{1 \leq j,k \leq n}$ with $c_{j,k} \in U_{j,k}$. After shrinking V_x we may assume $\psi_{j,k}(K_x) \subseteq U_{j,k}$ and $\psi_{j,\nu}(K_x) \subseteq U_{j,\nu}$. Hence ψ is contained in the set $\Theta(x) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \,|\, \tilde{\varphi}(K_x) \subseteq \overline{U}_x, \varphi_{j,k}(K_x) \subseteq \overline{U}_{j,k}, \varphi_{j,\nu}(K_x) \subseteq \overline{U}_{j,\nu}\}$, which contains an open neighbourhood of ψ. Since M is compact, M is covered by finitely many of the sets V_x, say $V_{x_1}, \ldots, V_{x_l}$. Then ψ is contained in $\Theta = \Theta(x_1) \cap \ldots \cap \Theta(x_l)$. We will now prove that Θ is sequentially compact:

Let φ_k be any sequence of automorphisms contained in Θ. Then, using Montel's theorem and passing to a subsequence, the sequence φ_k converges to a morphism $\varphi : \mathcal{M} \to \mathcal{M}$. It remains to show that φ is an automorphism of $\mathcal{M}$.

The underlying map $\tilde{\varphi} : M \to M$ is surjective since if $p \notin \tilde{\varphi}(M)$, then $\varphi \in \Delta(M, M \setminus \{p\})$ and therefore $\varphi_k \in \Delta(M, M \setminus \{p\})$ for k large enough which contradicts the assumption that φ_k is an automorphism. This also implies that there is an $x \in M$ such that the differential $D\tilde{\varphi}(x)$ is invertible. Using Hurwitz's theorem (see e.g. [Nar71], p. 80) it follows $\det(D\tilde{\varphi}(x)) \neq 0$ for all $x \in M$. Thus $\tilde{\varphi}$ is locally biholomorphic. Moreover, φ is locally invertible due to the special form of the sets $\Theta(x_i)$.

In order check that $\tilde{\varphi}$ is injective, let $p_1, p_2 \in M$, $p_1 \neq p_2$, such that $q = \tilde{\varphi}(p_1) = \tilde{\varphi}(p_2)$. Let Ω_j, $j = 1, 2$, be open neighbourhoods of p_j with $\Omega_1 \cap \Omega_2 = \emptyset$. By [Nar71], p. 79, Proposition 5, there exists k_0 with the property that $q \in \tilde{\varphi}_k(\Omega_1)$ and $q \in \tilde{\varphi}_k(\Omega_2)$ for all $k \geq k_0$. The bijectivity of the φ_k's now yields a contradiction to $\Omega_1 \cap \Omega_2 = \emptyset$. $\square$

Proposition 3.6. *The set* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is a topological group with composition of automorphisms as multiplication and inversion of automorphisms as the inverse.*

Proof. Let φ_k and ψ_l be two sequences of automorphisms of $\mathcal{M}$ converging to φ and ψ respectively. By the classical theory, $\tilde{\varphi}_k \circ \tilde{\psi}_l$ converges to $\tilde{\varphi} \circ \tilde{\psi}$, and $\tilde{\varphi}_k^{-1}$ to $\tilde{\varphi}^{-1}$. Let $U, V, W \subseteq M$ be open subsets with $\tilde{\varphi}(V) \subseteq W$, $\tilde{\varphi}_k(V) \subseteq W$, $\tilde{\psi}(U) \subseteq V$, $\tilde{\psi}_l(U) \subseteq V$, for k and l sufficiently large and let $f \in \mathcal{O}_{\mathcal{M}}(W)$. Then the sequence $(\varphi_k)^*(f) \in \mathcal{O}_{\mathcal{M}}(V)$ converges to $\varphi^*(f)$ on V, and by Lemma 3.4 $(\varphi_k \circ \psi_l)^*(f) = (\psi_l)^*((\varphi_k)^*(f))$ converges to $\psi^*(\varphi^*(f)) = (\varphi \circ \psi)^*(f)$ on U as $k, l \to \infty$, which shows that the multiplication is continuous. Consider now the inversion map $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $\varphi \mapsto \varphi^{-1}$. Let φ_k be a sequence in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ converging to $\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$. Note that since the automorphism group $\mathrm{Aut}(M)$ of the underlying manifold M is a topological group, the inversion map $\mathrm{Aut}(M) \to \mathrm{Aut}(M)$ is continuous. For any choice of local coordinate charts on $U, V \subseteq M$ such that the closure of $\tilde{\varphi}^{-1}(U)$ is contained in V we can conclude: Since $\tilde{\varphi}_k^{-1}$ converges to $\tilde{\varphi}^{-1}$, we have $\tilde{\varphi}_k^{-1}(U) \subseteq V$ for k sufficiently large. Identify $\mathcal{O}_{\mathcal{M}}(U) \cong \Gamma_{\Lambda E}(U)$, resp. $\mathcal{O}_{\mathcal{M}}(V) \cong \Gamma_{\Lambda E}(V)$ and decompose $\varphi^* = \varphi_0^* \exp(Y)$, $\varphi_k^* = \varphi_{k,0}^* \exp(Y_k)$ as in Section 2. Note that φ_0^* is induced by an automorphism φ_0 of the vector bundle E. We can verify by an observation in local coordinates that

the map $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(E)$, $\varphi \mapsto \varphi_0$, is continuous. Hence, the sequence $\varphi_{k,0}$ converges to φ_0 and $\varphi_{k,0}^*$ converges to φ_0^*. By [Mor58] the inversion on $\mathrm{Aut}(E)$ is continuous. Therefore, $(\varphi_{k,0}^{-1})^*$ converges to $(\varphi_0^{-1})^*$. Due to the finiteness of the logarithm and exponential series on nilpotent elements, Y_k converges to Y. Hence, $(\varphi_k^{-1})^* = \exp(-Y_k)(\varphi_{k,0}^*)^{-1}$ converges to $\exp(-Y)(\varphi_0^*)^{-1} = (\varphi^*)^{-1}$. $\hfill\square$

4. Non-existence of small subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$

In this section, we prove that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ does not contain small subgroups, which means that there exists an open neighbourhood of the identity in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ such that each subgroup contained in this neighbourhood consists only of the identity. As a consequence, the topological group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a real Lie group by a result of Yamabe (c.f. [Yam53]).

Before proving the non-existence of small subgroups, a few technical preparations are needed: Consider $\mathbb{C}^{m|n}$ and let $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ denote coordinates on $\mathbb{C}^{m|n}$. Let $U \subseteq \mathbb{C}^m$ be an open subset. For $f = \sum_\nu f_\nu \xi^\nu \in \mathcal{O}_{\mathbb{C}^{m|n}}(U)$ define

$$||f||_U = \left|\left| \sum_\nu f_\nu \xi^\nu \right|\right|_U := \sum_\nu ||f_\nu||_U \, ,$$

where $||f_\nu||_U$ denotes the supremum norm of the holomorphic function f_ν on U. For any morphism $\varphi : \mathcal{U} = (U, \mathcal{O}_{\mathbb{C}^{m|n}}|_U) \to \mathbb{C}^{m|n}$ define

$$||\varphi||_U := \sum_{i=1}^m ||\varphi^*(z_i)||_U + \sum_{j=1}^n ||\varphi^*(\xi_j)||_U.$$

Lemma 4.1. *Let $\mathcal{U} = (U, \mathcal{O}_{\mathbb{C}^{m|n}}|_U)$ be a superdomain in $\mathbb{C}^{m|n}$. For any relatively compact open subset U' of U there exists $\varepsilon > 0$ such that any morphism $\psi : \mathcal{U} \to \mathbb{C}^{m|n}$ with the property $||\psi - \mathrm{id}||_U < \varepsilon$ is biholomorphic as a morphism from $\mathcal{U}' = (U', \mathcal{O}_{\mathbb{C}^{m|n}}|_{U'})$ onto its image.*

Proof. Let $r > 0$ such that the closure of the polydisc $\Delta_r^n(z) = \{(w_1, \ldots, w_m) \mid |w_j - z_j| < r\}$ is contained in U for any $z = (z_1, \ldots, z_m) \in U'$. Let $v \in \mathbb{C}^m$ be any non-zero vector. Then we have $z + \zeta v \in U$ for any $z \in U'$ and ζ in the closure of $\Delta_{\frac{r}{||v||}}(0) = \{t \in \mathbb{C} \mid |t| < \frac{r}{||v||}\}$. If for given $\varepsilon > 0$ it is $||\psi - \mathrm{id}||_U < \varepsilon$ then we have in particular $||\tilde{\psi} - \mathrm{id}||_U < \varepsilon$ for the supremum norm of the underlying maps $\tilde{\psi}, \mathrm{id} : U \to \mathbb{C}^m$. Then, for the differential $D\tilde{\psi}$ of $\tilde{\psi}$ and any non-zero vector $v \in \mathbb{C}^m$ and any $z \in U'$ we have

$$\left|\left| D\tilde{\psi}(z)(v) - v \right|\right| = \left|\left| \frac{d}{dt}\left(\tilde{\psi}(z+tv) - (z+tv) \right) \right|\right| = \frac{1}{2\pi} \left|\left| \int_{\partial \Delta_{\frac{r}{||v||}}(0)} \frac{\tilde{\psi}(z+\zeta v) - (z+\zeta v)}{\zeta^2} d\zeta \right|\right|$$

$$\leq \frac{1}{2\pi} \int_{\partial \Delta_{\frac{r}{||v||}}(0)} \left|\left| \frac{\tilde{\psi}(z+\zeta v) - (z+\zeta v)}{\zeta^2} \right|\right| d\zeta$$

$$< \frac{\varepsilon ||v||}{r}.$$

This implies $||D\tilde{\psi}(z) - \mathrm{id}|| < \frac{\varepsilon}{r}$ with respect to the operator norm, for any $z \in U'$. Thus $\tilde{\psi}$ is locally biholomorphic on U' if ε is small enough. Moreover, ε might now be chosen such that $\tilde{\psi}$ is injective (see e.g. [Hir76], Chapter 2, Lemma 1.3).

Let $\psi_{j,k}, \psi_{j,\nu}$ be holomorphic functions on U such that $\psi^*(\xi_j) = \sum_{k=1}^n \psi_{j,k}\xi_k + \sum_{||\nu||\geq 3} \psi_{j,\nu}\xi^\nu$. It is now enough to show

$$\det((\psi_{j,k})_{1\leq j,k\leq n}(z)) \neq 0$$

for all $z \in U'$ and ε small enough in order to prove that ψ is a biholomorphism form $\mathcal{U}'$ onto its image. This follows from the fact that we assumed, via $||\psi - \mathrm{id}||_U < \varepsilon$, that $||\psi_{j,k}||_U < \varepsilon$ if $j \neq k$ and $||\psi_{j,j} - 1||_U < \varepsilon$. $\qquad\square$

This lemma now allows us to prove that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ contains no small subgroups; for a similar result in the classical case see [BM46], Theorem 1.

Proposition 4.2. *The topological group* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *has no small subgroups, i.e. there is a neighbourhood of the identity which contains no non-trivial subgroup.*

Proof. Let $U \subset V \subset W$ be open subsets of M such that U is relatively compact in V and V is relatively compact in W. Moreover, suppose that $\mathcal{W} = (W, \mathcal{O}_\mathcal{M}|_W)$ is isomorphic to a superdomain in $\mathbb{C}^{m|n}$ and let $z_1,\ldots,z_m,\xi_1,\ldots,\xi_n$ be local coordinates on $\mathcal{W}$. By definition $\Delta(\overline{V}, W) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})| \tilde{\varphi}(\overline{V}) \subseteq W\}$ and $\Delta(\overline{U}, V)$ are open neighbourhoods of the identity in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. Choose $\varepsilon > 0$ as in the preceding lemma such that any morphism $\chi : \mathcal{V} \to \mathbb{C}^{m|n}$ with $||\chi - \mathrm{id}||_V < \varepsilon$ is biholomorphic as a morphism from $\mathcal{U}$ onto its image. Let $\Omega \subseteq \Delta(\overline{V}, W) \cap \Delta(\overline{U}, V)$ be the subset whose elements φ satisfy $||\varphi - \mathrm{id}||_V < \varepsilon$. The set Ω is open and contains the identity. Since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is locally compact by Lemma 3.5, it is enough to show that each compact subgroup $Q \subseteq \Omega$ is trivial. Otherwise for non-compact Q, let Ω' be an open neighbourhood of the identity with compact closure $\overline{\Omega}'$ which is contained in Ω, and suppose $Q \subseteq \Omega'$. Then $\overline{Q} \subseteq \overline{\Omega}' \subset \Omega$ is a compact subgroup, and Q is trivial if $\overline{Q}$ is trivial.

Define a morphism $\psi : \mathcal{V} \to \mathbb{C}^{m|n}$ by setting

$$\psi^*(z_i) = \int_Q q^*(z_i)\, dq \quad \text{and} \quad \psi^*(\xi_j) = \int_Q q^*(\xi_j)\, dq,$$

where the integral is taken with respect to the normalized Haar measure on Q. This yields a holomorphic morphism $\psi : \mathcal{V} \to \mathbb{C}^{m|n}$ since each $q \in Q$ defines a holomorphic morphism $\mathcal{V} \to \mathcal{W} \subseteq \mathbb{C}^{m|n}$. Its underlying map is $\tilde{\psi}(z) = \int_Q \tilde{q}(z)\, dq$. The morphism ψ satisfies

$$||\psi^*(z_i) - z_i||_V = \left|\left|\int_Q (q^*(z_i) - z_i)\, dq\right|\right|_V \leq \int_Q ||q^*(z_i) - z_i||_V\, dq$$

and similarly

$$||\psi^*(\xi_j) - \xi_j||_V \leq \int_Q ||q^*(\xi_j) - \xi_j||_V\, dq.$$

Consequently, we have

$$||\psi - \mathrm{id}||_V = \sum_{i=1}^m ||\psi^*(z_i) - z_i||_V + \sum_{j=1}^n ||\psi^*(\xi_j) - \xi_j||_V$$

$$\leq \int_Q \left(\sum_{i=1}^m ||q^*(z_i) - z_i||_V + \sum_{j=1}^n ||q^*(\xi_j) - \xi_j||_V\right) dq$$

$$= \int_Q ||q - \mathrm{id}||_V\, dq < \varepsilon.$$

Thus by the preceding lemma, $\psi|_U$ is a biholomorphic morphism onto its image. Furthermore, on U we have $\psi \circ q' = \psi$ for any $q' \in Q$ since

$$(\psi \circ q')^*(z_i) = (q')^*(\psi^*(z_i)) = (q')^* \left(\int_Q q^*(z_i)\, dq \right) = \int_Q (q')^*(q^*(z_i))\, dq$$

$$= \int_Q (q \circ q')^*(z_i)\, dq = \int_Q q^*(z_i)\, dq = \psi^*(z_i)$$

due to the invariance of the Haar measure, and also

$$(\psi \circ q')^*(\xi_j) = \psi^*(\xi_j).$$

The equality $\psi \circ q' = \psi$ on U implies $q'|_U = \mathrm{id}_{\mathcal{U}}$ because of the invertibility of ψ. By the identity principle it follows that $q' = \mathrm{id}_{\mathcal{M}}$ if M is connected, and hence $Q = \{\mathrm{id}_{\mathcal{M}}\}$.

In general, M has only finitely many connected components since M is compact. Therefore, a repetition of the preceding argument yields the existence of a neighbourhood of the identity of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ without any non-trivial subgroups. $\qquad\square$

By Theorem 3 in [Yam53], the preceding proposition implies the following:

Corollary 4.3. *The topological group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ can be endowed with the structure of a real Lie group.*

5. ONE-PARAMETER SUBGROUPS OF $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$

In order to obtain results on the regularity of the action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on the compact complex supermanifold $\mathcal{M}$ and to characterize the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, we study continuous one-parameter subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. Each continuous one-parameter subgroup $\mathbb{R} \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is an analytic map between the Lie groups $\mathbb{R}$ and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$.

We prove that the action of each continuous one-parameter subgroup of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is analytic and induces an even holomorphic super vector field on $\mathcal{M}$. Consequently, the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ may be identified with the Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even holomorphic super vector fields on $\mathcal{M}$, and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group whose action on the supermanifold $\mathcal{M}$ is holomorphic.

Definition 5.1. A continuous one-parameter subgroup of automorphisms of $\mathcal{M}$ is a family of automorphisms $\varphi_t : \mathcal{M} \to \mathcal{M}$, $t \in \mathbb{R}$ such that the map $\mathbb{R} \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $t \mapsto \varphi_t$, is a continuous group homomorphism.

Remark 5.2. Let $\varphi_t : \mathcal{M} \to \mathcal{M}$, $t \in \mathbb{R}$, be a family of automorphisms satisfying $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$, and such that $\tilde{\varphi} : \mathbb{R} \times M \to M$, $\tilde{\varphi}(t, p) = \tilde{\varphi}_t(p)$ is continuous. Then φ_t is a continuous one-parameter subgroup if and only if the following condition is satisfied: Let $U, V \subset M$ be open subsets, and $[a, b] \subset \mathbb{R}$ such that $\tilde{\varphi}([a, b] \times U) \subseteq V$. Assume moreover that there are local coordinates $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ for $\mathcal{M}$ on U. Then for any $f \in \mathcal{O}_{\mathcal{M}}(V)$ there are continuous functions $f_\nu : [a, b] \times U \to \mathbb{C}$ with $(f_\nu)_t = f_\nu(t, \cdot) \in \mathcal{O}_{\mathcal{M}}(U)$ for fixed $t \in [a, b]$ such that

$$(\varphi_t)^*(f) = \sum_\nu f_\nu(t, z)\xi^\nu.$$

We say that the action of the one-parameter subgroup φ on $\mathcal{M}$ is analytic if each $f_\nu(t, z)$ is analytic in both components.

Proposition 5.3. *Let φ be continuous one-parameter subgroup of automorphisms on $\mathcal{M}$. Then the action of φ on $\mathcal{M}$ is analytic.*

Remark 5.4. Defining a continuous one-parameter subgroup as in Remark 5.2, the statement of Proposition 5.3 also holds true for complex supermanifolds $\mathcal{M}$ with non-compact underlying manifold M as compactness of M is not needed for the proof.

For the proof of the proposition the following technical lemma is needed:

Lemma 5.5. *Let $U \subseteq V \subseteq \mathbb{C}^m$ be open subsets, $p \in U$, $\Omega \subseteq \mathbb{R}$ an open connected neighbourhood of 0, and let $\alpha : \Omega \times U \to V$ be a continuous map satisfying $\alpha(t, z) = \alpha(t+s, z) - f(t, s, z)$ for some continuous function f which is analytic in (t, z) and small s, t, and z near p. If α is holomorphic in the second component, then it is analytic on a neighbourhood of $(0, p)$.*

Proof. For small $t, h > 0$, z near p, we have

$$
\begin{aligned}
h \cdot \alpha(t, z) &= \int_0^h \alpha(t + s, z)ds - \int_0^h f(t, s, z)ds \\
&= \int_t^{h+t} \alpha(s, z)ds - \int_0^h \alpha(s, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds \\
&= \int_h^{h+t} \alpha(s, z)ds - \int_0^t \alpha(s, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds \\
&= \int_0^t (\alpha(s + h, z) - \alpha(s, z))ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds \\
&= \int_0^t f(s, h, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds.
\end{aligned}
$$

The assumption that f is a continuous function which is analytic in the first and third component therefore implies that α is analytic. $\square$

Proof of Proposition 5.3. Due to the action property $\varphi_{s+t} = \varphi_s \circ \varphi_t$ it is enough to show the statement for the restriction of φ to $(-\varepsilon, \varepsilon) \times \mathcal{M}$ for some $\varepsilon > 0$. Let $U, V \subseteq M$ be open subsets such that U is relatively compact in V, and such that there are local coordinates $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ on V for $\mathcal{M}$. Choose $\varepsilon > 0$ such that $\tilde{\varphi}_t(U) \subseteq V$ for any $t \in (-\varepsilon, \varepsilon)$. Let $\alpha_{i,\nu}$, $\beta_{j,\nu}$ be continuous functions on $(-\varepsilon, \varepsilon) \times U$ with $(\varphi_t)^*(z_i) = \sum_{|\nu|=0} \alpha_{i,\nu}(t, z)\xi^\nu$ and $(\varphi_t)^*(\xi_j) = \sum_{|\nu|=1} \beta_{j,\nu}(t, z)\xi^\nu$, where $|\nu| = |(\nu_1, \ldots, \nu_n)| = (\nu_1 + \ldots + \nu_n) \bmod 2 \in \mathbb{Z}_2$. We have to show that α and β are analytic in (t, z). The induced map $\psi' : (-\varepsilon, \varepsilon) \times U \times \mathbb{C}^n \to V \times \mathbb{C}^n$ on the underlying vector bundle is given by

$$
\left(t, \begin{pmatrix} z_1 \\ \vdots \\ z_m \\ v_1 \\ \vdots \\ v_n \end{pmatrix} \right) \mapsto \begin{pmatrix} \alpha_{1,0}(t, z) \\ \vdots \\ \alpha_{m,0}(t, z) \\ \sum_{k=1}^n \beta_{1,k}(t, z)v_k \\ \vdots \\ \sum_{k=1}^n \beta_{n,k}(t, z)v_k \end{pmatrix},
$$

where $\beta_{j,k} = \beta_{j,e_k}$ if $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the k-th unit vector. The map ψ' is a local continuous one-parameter subgroup on $U \times \mathbb{C}^n$ because φ is a continuous one-parameter subgroup. By a result of Bochner and Montgomery the map ψ' is analytic in (t, z, v) (see [BM45], Theorem 4). Hence, the map $\psi : (-\varepsilon, \varepsilon) \times \mathcal{U} \to \mathcal{V}$ given by $(\psi_t)^*(z_i) = \alpha_i(t, z)$, $(\psi_t)^*(\xi_j) = \sum_{k=1}^n \beta_{j,k}(t, z)\xi_k$ is analytic. Let X be the local vector field on $\mathcal{U}$ induced by ψ, i.e.

$$
X(f) = \left. \frac{\partial}{\partial t} \right|_0 (\psi_t)^*(f).
$$

We may assume that X is non-degenerate, i.e. the evaluation of X in p, $X(p)$, does not vanish for all $p \in U$. Otherwise, consider, instead of φ, the diagonal action on $\mathbb{C} \times \mathcal{M}$ acting by addition of t in the first component and φ_t in the second, and note that this action is analytic precisely if φ is analytic. For the differential $d\psi$ of ψ in $(0,p)$ we have

$$d\psi \left(\left.\frac{\partial}{\partial t}\right|_{(0,p)} \right) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} \circ \psi^* = X(p) \neq 0.$$

Therefore, the restricted map $\psi|_{(-\varepsilon,\varepsilon) \times \{p\}}$ is an immersion and its image $\psi((-\varepsilon,\varepsilon) \times \{p\})$ is a subsupermanifold of $\mathcal{V}$. Let $\mathcal{S}$ be a subsupermanifold of $\mathcal{U}$ transversal to $\psi((-\varepsilon,\varepsilon) \times \{p\})$ in p. The map $\psi|_{(-\varepsilon,\varepsilon) \times \mathcal{S}}$ is a submersion in $(0,p)$ since $d\psi(T_{(0,p)}(-\varepsilon,\varepsilon) \times \{p\})) = T_p\psi((-\varepsilon,\varepsilon) \times \{p\})$ and $d\psi(T_{(0,p)}\{0\} \times \mathcal{S}) = T_p\mathcal{S}$ because $\psi|_{\{0\} \times \mathcal{U}} = \mathrm{id}$. Hence $\chi := \psi|_{(-\varepsilon,\varepsilon) \times \mathcal{S}}$ is locally invertible around $(0,p)$, and thus invertible as a map onto its image after possibly shrinking U and ε, and

$$\chi_* \left(\frac{\partial}{\partial t} \right) = (\chi^{-1})^* \circ \frac{\partial}{\partial t} \circ \chi^* = (\chi^{-1})^* \circ \chi^* \circ X = X.$$

Therefore, after defining new coordinates $w_1, \ldots, w_m, \theta_1, \ldots, \theta_n$ for $\mathcal{M}$ on U via χ, we have $X = \frac{\partial}{\partial w_1}$ and $(\varphi_t)^*$ is of the form

$$(\varphi_t)^*(w_1) = w_1 + t + \sum_{|\nu|=0, \nu \neq 0} \alpha_{1,\nu}(t,w)\theta^\nu,$$

$$(\varphi_t)^*(w_i) = w_i + \sum_{|\nu|=0, \nu \neq 0} \alpha_{i,\nu}(t,w)\theta^\nu \quad \text{for } i \neq 1,$$

$$(\varphi_t)^*(\theta_j) = \theta_j + \sum_{|\nu|=1, ||\nu|| \neq 1} \beta_{j,\nu}(t,w)\theta^\nu,$$

for appropriate $\alpha_{i,\nu}$, $\beta_{j,\nu}$, where $||\nu|| = ||(\nu_1, \ldots, \nu_n)|| = \nu_1 + \ldots + \nu_n$.
For small s and t we have

$$\varphi_t^* \left(\varphi_s^*(w_i) \right) = \varphi_t^* \left(w_i + \delta_{1,i}s + \sum_{|\nu|=0, ||\nu|| \neq 0} \alpha_{i,\nu}(s,w)\theta^\nu \right)$$

$$= w_i + \delta_{i,1}(t+s) + \sum_{|\nu|=0, ||\nu|| \neq 0} \alpha_{i,\nu}(t,w)\theta^\nu + \sum_{|\nu|=0, ||\nu|| \neq 0} \varphi_t^*(\alpha_{i,\nu}(s,w)\theta^\nu). \qquad (1)$$

Let $f_{i,\nu}(t,s,w)$ be such that

$$\sum_{|\nu|=0, ||\nu|| \neq 0} \varphi_t^*(\alpha_{i,\nu}(s,w)\theta^\nu) = \sum_{|\nu|=0, ||\nu|| \neq 0} f_{i,\nu}(t,s,w)\theta^\nu. \qquad (2)$$

For fixed ν_0 the coefficient of θ^{ν_0}, $f_{i,\nu_0}(t,s,w)$, depends only on $\alpha_{i,\nu_0}(s, w+te_1)$, $\beta_{j,\mu}(t,w)$ for μ with $||\mu|| \leq ||\nu_0|| - 1$, and $\alpha_{j,\nu}(t,w)$ and its partial derivatives in the second component for ν with $||\nu|| \leq ||\nu_0|| - 2$. This can be shown by a calculation using the special form of $\varphi_t^*(w_j)$ and $\varphi_t^*(\theta_j)$ and general properties of the pullback of a morphism of supermanifolds. Assume now that the analyticity near $(0,p)$ of $\alpha_{i,\nu}$, $\beta_{j,\mu}$ is shown for $||\nu||, ||\mu|| < 2k$ and all i,j. Let ν_0 be such that $||\nu_0|| = 2k$. Then $f_{i,\nu_0}(t,s,w)$ is a continuous function which is analytic in (t,w) near $(0,p)$ for fixed s. Since $\varphi_t^*(\varphi_s^*(w_i)) = \varphi_{t+s}^*(w_i)$, using (1) and (2) we get

$$\alpha_{i,\nu_0}(t,w) + f_{i,\nu_0}(t,s,w) = \alpha_{i,\nu_0}(t+s,w),$$

and thus $\alpha_{i,\nu_0}(t,w)$ is analytic near $(0,p)$ by Lemma 5.5. Similarly, it can be shown that β_{j,μ_0} is analytic for $||\mu_0|| = 2k+1$ if $\alpha_{i,\nu}$, $\beta_{j,\mu}$ for $||\nu||, ||\mu|| < 2k+1$. $\qquad \square$

Corollary 5.6. *The Lie algebra of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is isomorphic to the Lie algebra* $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ *of even super vector fields on* $\mathcal{M}$, *and* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is a complex Lie group.*

Proof. If $\gamma : \mathbb{R} \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $t \mapsto \gamma_t$ is a continuous one-parameter subgroup, then by Proposition 5.3 the action of φ on $\mathcal{M}$ is analytic. Therefore, γ induces an even holomorphic super vector field $X(\gamma)$ on $\mathcal{M}$ by setting

$$X(\gamma) = \left. \frac{\partial}{\partial t} \right|_0 (\gamma_t)^*,$$

and γ is the flow map of $X(\gamma)$. On the other hand, each $X \in \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ is globally integrable since M is compact (c.f [GW13], Theorem 5.4). Its flow defines a one-parameter subgroup γ^X of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, which is continuous. This yields an isomorphism of Lie algebras

$$\mathrm{Lie}(\mathrm{Aut}_{\bar{0}}(\mathcal{M})) \to \mathrm{Vec}_{\bar{0}}(\mathcal{M}).$$

Consequently, we have $\mathrm{Lie}(\mathrm{Aut}_{\bar{0}}(\mathcal{M})) \cong \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ and since $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ is a complex Lie algebra, $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group. $\square$

The Lie group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ naturally acts on $\mathcal{M}$; this action $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ is given by $\mathrm{ev}_g \circ \psi^* = g^*$ where ev_g denotes the evaluation in $g \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ in the first component.

Corollary 5.7. *The natural action of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *on* $\mathcal{M}$ *defines a holomorphic morphism of supermanifolds* $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$.

Proof. Since the action of each continuous one-parameter subgroup of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is holomorphic by the preceding considerations, and each $g \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a biholomorphic morphism $g : \mathcal{M} \to \mathcal{M}$, the action Φ is a holomorphic. $\square$

If a Lie supergroup $\mathcal{G}$ (with Lie superalgebra $\mathfrak{g}$ of right-invariant super vector fields) acts on a supermanifold $\mathcal{M}$ via $\psi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, this action ψ induces an infinitesimal action $d\psi : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ defined by $d\psi(X) = (X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \psi^*$ for any $X \in \mathfrak{g}$, where $X \otimes \mathrm{id}_{\mathcal{M}}^*$ denotes the canonical extension of the vector field X on $\mathcal{G}$ to a vector field on $\mathcal{G} \times \mathcal{M}$, and $(X(e) \otimes \mathrm{id}_{\mathcal{M}}^*)$ is its evaluation in the neutral element e of $\mathcal{G}$.

Corollary 5.8. *Identifying the Lie algebra of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *with* $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ *as in Corollary 5.6, the induced infinitesimal action of the action* $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ *in Corollary 5.7 is the inclusion* $\mathrm{Vec}_{\bar{0}}(\mathcal{M}) \hookrightarrow \mathrm{Vec}(\mathcal{M})$.

6. The Lie superalgebra of vector fields

In this section, we prove that the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of holomorphic super vector fields on a compact complex supermanifold $\mathcal{M}$ is finite-dimensional.

First, we prove that $\mathrm{Vec}(\mathcal{M})$ is finite-dimensional if $\mathcal{M}$ is a split supermanifold using that its tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is a coherent sheaf of $\mathcal{O}_M$-modules, where $\mathcal{O}_M$ denotes again the sheaf of holomorphic functions on the underlying manifold M. Then the statement in the general case is deduced using a filtration of the tangent sheaf.

Remark that since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a complex Lie group with Lie algebra isomorphic to the Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even holomorphic super vector fields on $\mathcal{M}$ (see Corollary 5.6), we already know that the even part of $\mathrm{Vec}(\mathcal{M}) = \mathrm{Vec}_{\bar{0}}(\mathcal{M}) \oplus \mathrm{Vec}_{\bar{1}}(\mathcal{M})$ is finite-dimensional.

Lemma 6.1. *Let* $\mathcal{M}$ *be a split complex supermanifold. Then its tangent sheaf* $\mathcal{T}_{\mathcal{M}}$ *is a coherent sheaf of* $\mathcal{O}_M$-modules.

Proof. Since $\mathcal{M}$ is split, its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$ as an $\mathcal{O}_M$-module, where $\mathcal{E}$ is the sheaf of sections of a holomorphic vector bundle on the underlying manifold M. Thus, the structure sheaf $\mathcal{O}_{\mathcal{M}}$, and hence also the tangent sheaf $\mathcal{T}_{\mathcal{M}}$, carry the structure of a sheaf of $\mathcal{O}_M$-modules. Let $U \subset M$ be an open subset such that there exist even coordinates $z_1, \ldots, z_m$ and odd coordinates $\xi_1, \ldots, \xi_n$. Any derivation $D \in \mathcal{T}_{\mathcal{M}}(U)$ on U can uniquely be written as

$$D = \sum_{\nu \in (\mathbb{Z}_2)^n} \left(\sum_{i=1}^{m} f_{i,\nu}(z) \xi^\nu \frac{\partial}{\partial z_i} + \sum_{j=1}^{n} g_{j,\nu}(z) \xi^\nu \frac{\partial}{\partial \xi_j} \right)$$

where $f_{i,\nu}$, $g_{j,\nu}$ are holomorphic functions on U. Therefore, the restricted sheaf $\mathcal{T}_{\mathcal{M}}|_U$ is isomorphic to $(\mathcal{O}_M|_U)^{2^n(m+n)}$ and $\mathcal{T}_{\mathcal{M}}$ is coherent over $\mathcal{O}_M$. $\qquad\square$

Proposition 6.2. *The Lie superalgebra* $\mathrm{Vec}(\mathcal{M})$ *of holomorphic super vector fields on a compact complex supermanifold* $\mathcal{M}$ *is finite-dimensional.*

Proof. First, assume that $\mathcal{M}$ is split. Then the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is a coherent sheaf of $\mathcal{O}_M$-modules. Thus, the space of global sections of $\mathcal{T}_{\mathcal{M}}$, $\mathrm{Vec}(\mathcal{M}) = \mathcal{T}_{\mathcal{M}}(M)$, is finite-dimensional since M is compact (c.f. [CS53]).

Now, let $\mathcal{M}$ be an arbitrary compact complex supermanifold. We associate the split complex supermanifold $\mathrm{gr}\,\mathcal{M} = (M, \mathrm{gr}\,\mathcal{O}_{\mathcal{M}})$ as described in Section 2. Let $\mathcal{I}_{\mathcal{M}}$ denote as before the subsheaf of ideal in $\mathcal{O}_{\mathcal{M}}$ generated by the odd elements. Define the filtration of sheaves of Lie superalgebras

$$\mathcal{T}_{\mathcal{M}} =: (\mathcal{T}_{\mathcal{M}})_{(-1)} \subset (\mathcal{T}_{\mathcal{M}})_{(0)} \subset (\mathcal{T}_{\mathcal{M}})_{(1)} \subset \ldots \subset (\mathcal{T}_{\mathcal{M}})_{(n+1)} = 0$$

of the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ by setting

$$(\mathcal{T}_{\mathcal{M}})_{(k)} = \{ D \in \mathcal{T}_{\mathcal{M}} | \, D(\mathcal{O}_{\mathcal{M}}) \subset (\mathcal{I}_{\mathcal{M}})^k, \, D(\mathcal{I}_{\mathcal{M}}) \subset (\mathcal{I}_{\mathcal{M}})^{k+1} \}$$

for $k \geq 0$. Moreover, define $\mathrm{gr}_k(\mathcal{T}_{\mathcal{M}}) = (\mathcal{T}_{\mathcal{M}})_{(k)}/(\mathcal{T}_{\mathcal{M}})_{(k+1)}$ and set

$$\mathrm{gr}(\mathcal{T}_{\mathcal{M}}) = \bigoplus_{k \geq -1} \mathrm{gr}_k(\mathcal{T}_{\mathcal{M}}).$$

By [Oni98], Proposition 1, the sheaf $\mathrm{gr}(\mathcal{T}_{\mathcal{M}})$ is isomorphic to the tangent sheaf of the associated split supermanifold $\mathrm{gr}\,\mathcal{M}$. By the preceding considerations, the space of holomorphic super vector fields on $\mathrm{gr}\,\mathcal{M}$,

$$\mathrm{Vec}(\mathrm{gr}\,\mathcal{M}) = \mathrm{gr}(\mathcal{T}_{\mathcal{M}})(M) = \bigoplus_{k \geq -1} \mathrm{gr}_k(\mathcal{T}_{\mathcal{M}})(M),$$

is of finite dimension. The projection onto the quotient yields

$$\dim(\mathcal{T}_{\mathcal{M}})_{(k)}(M) - \dim(\mathcal{T}_{\mathcal{M}})_{(k+1)}(M) \leq \dim(\mathrm{gr}_k(\mathcal{T}_{\mathcal{M}})(M))$$

and $\dim(\mathcal{T}_{\mathcal{M}})_{(n)}(M) = \dim(\mathrm{gr}_n(\mathcal{T}_{\mathcal{M}})(M))$ and hence by induction

$$\dim(\mathcal{T}_{\mathcal{M}})_{(k)}(M) \leq \sum_{j \geq k} \dim(\mathrm{gr}_j(\mathcal{T}_{\mathcal{M}})(M)),$$

which gives

$$\dim(\mathcal{T}_{\mathcal{M}}(M)) = \dim\left((\mathcal{T}_{\mathcal{M}})_{(-1)}(M)\right) \leq \dim\left(\mathrm{gr}(\mathcal{T}_{\mathcal{M}})(M)\right).$$

In particular, $\dim(\mathcal{T}_{\mathcal{M}}(M))$ is finite. $\qquad\square$

Remark 6.3. The proof of the preceding proposition also shows the following inequality:

$$\dim(\mathrm{Vec}(\mathcal{M})) \leq \dim(\mathrm{Vec}(\mathrm{gr}\,\mathcal{M}))$$

7. The automorphism group

In this section, the automorphism group of a compact complex supermanifold is defined. This is done via the formalism of Harish-Chandra pairs for complex Lie supergroups (c.f. [Vis11]). The underlying classical Lie group is $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and the Lie superalgebra is $\mathrm{Vec}(\mathcal{M})$, the Lie superalgebra of super vector fields on $\mathcal{M}$. Moreover, we prove that the automorphism group satisfies a universal property.

Consider the representation α of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$ given by

$$\alpha(g)(X) = g_*(X) = (g^{-1})^* \circ X \circ g^* \quad \text{for} \quad g \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}),\, X \in \mathrm{Vec}(\mathcal{M}).$$

This representation α preserves the parity on $\mathrm{Vec}(\mathcal{M})$, and its restriction to $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ coincides with the adjoint action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on its Lie algebra $\mathrm{Lie}(\mathrm{Aut}_{\bar{0}}(\mathcal{M})) \cong \mathrm{Vec}_{\bar{0}}(\mathcal{M})$. Moreover, the differential $(d\alpha)_{\mathrm{id}}$ at the identity $\mathrm{id} \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is the adjoint representation of $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$:
Let X and Y be super vector fields on $\mathcal{M}$. Assume that X is even and let φ^X denote the corresponding one-parameter subgroup. Then we have

$$(d\alpha)_{\mathrm{id}}(X)(Y) = \left.\frac{\partial}{\partial t}\right|_0 (\varphi^X_t)_*(Y) = [X,Y];$$

see e.g. [Ber14], Corollary 3.8. Therefore, the pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$ together with the representation α is a complex Harish-Chandra pair, and using the equivalence between the category of complex Harish-Chandra pairs and complex Lie supergroups (c.f. [Vis11], § 2), we can define the automorphism group of a compact complex supermanifold $\mathcal{M}$ as follows:

Definition 7.1. Define the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a compact complex supermanifold to be the unique complex Lie supergroup associated to the Harish-Chandra pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$ with adjoint representation α.

Since the action $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ induces the inclusion $\mathrm{Vec}_{\bar{0}}(\mathcal{M}) \hookrightarrow \mathrm{Vec}(\mathcal{M})$ as infinitesimal action (see Corollary 5.8), there exists a Lie supergroup action $\Psi : \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ with the identity $\mathrm{Vec}(\mathcal{M}) \to \mathrm{Vec}(\mathcal{M})$ as induced infinitesimal action and $\Psi|_{\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M}} = \psi$ (c.f. Theorem 5.35 in [Ber14]).
The automorphism group together with Ψ satisfies a universal property:

Theorem 7.2. *Let $\mathcal{G}$ be a complex Lie supergroup with a holomorphic action $\Psi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$. Then there is a unique morphism $\sigma : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups such that the diagram*

is commutative.

Proof. Let G be the underlying Lie group of $\mathcal{G}$. For each $g \in G$, we have a morphism $\Psi_{\mathcal{G}}(g) : \mathcal{M} \to \mathcal{M}$ by setting $(\Psi_{\mathcal{G}}(g))^* = \mathrm{ev}_g \circ (\Psi_{\mathcal{G}})^*$. This morphism $\Psi_{\mathcal{G}}(g)$ is an automorphism of $\mathcal{M}$ with inverse $\Psi_{\mathcal{G}}(g^{-1})$ and gives rise to a group homomorphism $\tilde{\sigma} : G \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $g \mapsto \Psi_{\mathcal{G}}(g)$.
Let $\mathfrak{g}$ denote the Lie superalgebra (of right-invariant super vector fields) of $\mathcal{G}$, and $d\Psi_{\mathcal{G}} : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ the infinitesimal action induced by $\Psi_{\mathcal{G}}$. The restriction of $d\Psi_{\mathcal{G}}$ to the even part $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$ of $\mathfrak{g}$ coincides with the differential $(d\tilde{\sigma})_e$ of $\tilde{\sigma}$ at the identity $e \in G$.

Moreover, if $\alpha_{\mathcal{G}}$ denotes the adjoint action of G on $\mathfrak{g}$, and α denotes, as before, the adjoint action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$, we have

$$d\Psi_{\mathcal{G}}(\alpha_{\mathcal{G}}(g)(X)) = (\Psi_{\mathcal{G}}(g^{-1}))^* \circ d\Psi_{\mathcal{G}}(X) \circ (\Psi_{\mathcal{G}}(g))^* = (\tilde{\sigma}(g^{-1}))^* \circ d\Psi_{\mathcal{G}}(X) \circ (\tilde{\sigma}(g))^*$$
$$= \alpha(\tilde{\sigma}(g))(d\Psi_{\mathcal{G}}(X))$$

for any $g \in G$, $X \in \mathfrak{g}$. Using the correspondence between Lie supergroups and Harish-Chandra pairs, it follows that there is a unique morphism $\sigma : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups with underlying map $\tilde{\sigma}$ and derivative $d\Psi_{\mathcal{G}} : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ (see e.g. [Vis11], § 2), and σ satisfies $\Psi \circ (\sigma \times \mathrm{id}_{\mathcal{M}}) = \Psi_{\mathcal{G}}$.

The uniqueness of σ follows from the fact that each morphism $\tau : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups fulfilling the same properties as σ necessarily induces the map $d\Psi_{\mathcal{G}} : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ on the level of Lie superalgebras and its underlying map $\tilde{\tau}$ has to satisfy $\tilde{\tau}(g) = \Psi_{\mathcal{G}}(g) = \tilde{\sigma}(g)$. $\qquad\square$

Remark 7.3. Since the morphism σ in Theorem 7.2 is unique, the automorphism group of a compact complex supermanifold $\mathcal{M}$ is the unique Lie supergroup satisfying the universal property formulated in Theorem 7.2.

Remark 7.4. We say that a real Lie supergroup $\mathcal{G}$ acts on $\mathcal{M}$ by holomorphic transformations if the underlying Lie group G acts on the complex manifold M by holomorphic transformations and if there is a homomorphism of Lie superalgebras $\mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ which is compatible with the action of G on M. Using the theory of Harish-Chandra pairs, we also have the Lie supergroup $\mathcal{G}^{\mathbb{C}}$, the universal complexification of $\mathcal{G}$; see [Kal15]. The underlying Lie group of $\mathcal{G}^{\mathbb{C}}$ is the universal complexification $G^{\mathbb{C}}$ of the Lie group G. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ denote the Lie superalgebra of $\mathcal{G}$, $\mathfrak{g}_{\bar{0}}$ the Lie algebra of G. Then the Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ is a quotient of $\mathfrak{g}_{\bar{0}} \otimes \mathbb{C}$, and the Lie superalgebra of $\mathcal{G}^{\mathbb{C}}$ can be realised as $\mathfrak{g}_{\bar{0}}^{\mathbb{C}} \oplus (\mathfrak{g}_{\bar{1}} \otimes \mathbb{C})$. The action of G on $\mathcal{M}$ extends to a holomorphic $G^{\mathbb{C}}$-action on $\mathcal{M}$, and the homomorphism $\mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ extends to a homomorphism $\mathfrak{g}_{\bar{0}}^{\mathbb{C}} \oplus (\mathfrak{g}_{\bar{1}} \otimes \mathbb{C}) \to \mathrm{Vec}(\mathcal{M})$ of complex Lie superalgebras, which is compatible with the $G^{\mathbb{C}}$-action on $\mathcal{M}$. Thus, we have a holomorphic $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{M}$ extending the $\mathcal{G}$-action. Moreover, there is a morphism $\sigma : \mathcal{G}^{\mathbb{C}} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups as in Theorem 7.2.

Example 7.5. Let $\mathcal{M} = \mathbb{C}^{0|1}$. Denoting the odd coordinate on $\mathbb{C}^{0|1}$ by ξ, each super vector field on $\mathbb{C}^{0|1}$ is of the form $X = a\xi\frac{\partial}{\partial \xi} + b\frac{\partial}{\partial \xi}$ for $a, b \in \mathbb{C}$. The flow $\varphi : \mathbb{C} \times \mathcal{M} \to \mathcal{M}$ of $a\xi\frac{\partial}{\partial \xi}$ is given by $(\varphi_t)^*(\xi) = e^{at}\xi$, and the flow $\psi : \mathbb{C}^{0|1} \times \mathcal{M} \to \mathcal{M}$ of $b\frac{\partial}{\partial \xi}$ by $\psi^*(\xi) = b\tau + \xi$. Let $X_0 = \xi\frac{\partial}{\partial \xi}$ and $X_1 = \frac{\partial}{\partial \xi}$. Then $\mathrm{Vec}(\mathbb{C}^{0|1}) = \mathbb{C}X_0 \oplus \mathbb{C}X_1 = \mathbb{C}^{1|1}$, where the Lie algebra structure on $\mathbb{C}^{1|1}$ is given by $[X_0, X_1] = -X_1$ and $[X_1, X_1] = 0$. Note that this Lie superalgebra is isomorphic to the Lie superalgebra of right-invariant vector fields on the Lie supergroup $(\mathbb{C}^{1|1}, \mu_{0,1})$, where the multiplication $\mu = \mu_{0,1}$ is given by $\mu^*(t) = t_1 + t_2$ and $\mu^*(\tau) = \tau_1 + e^{t_1}\tau_2$; for the Lie supergroup structures on $\mathbb{C}^{1|1}$ see e.g. [GW13], Lemma 3.1. In particular, the Lie superalgebra $\mathrm{Vec}(\mathbb{C}^{0|1})$ is not abelian.

Since each automorphism φ of $\mathbb{C}^{0|1}$ is given by $\varphi^*(\xi) = c \cdot \xi$ for some $c \in \mathbb{C}$, $c \neq 0$, we have $\mathrm{Aut}_{\bar{0}}(\mathbb{C}^{0|1}) \cong \mathbb{C}^*$.

8. The case of a superdomain with bounded underlying domain

In the classical case, the automorphism group of a bounded domain $U \subset \mathbb{C}^m$ is a (real) Lie group (see Theorem 13 in "Sur les groupes de transformations analytiques" in [Car79]). If $\mathcal{U} \subset \mathbb{C}^{m|n}$ is a superdomain whose underlying set U is a bounded domain in $\mathbb{C}^m$, it is in general not possible to endow its set of automorphisms with the structure of a Lie group such

that the action on $\mathcal{U}$ is smooth, as will be illustrated in an example. In particular, there is no Lie supergroup satisfying the universal property as the automorphism group of a compact complex supermanifold $\mathcal{M}$ does as formulated in Theorem 7.2.

Example 8.1. Consider the superdomain $\mathcal{U}$ of dimension $(1|2)$ with bounded underlying domain $U \subset \mathbb{C}$. Let z, θ_1, θ_2 denote coordinates for $\mathcal{M}$. For any holomorphic function f on U, define the even super vector field $X_f = f(z)\theta_1\theta_2\frac{\partial}{\partial z}$. The reduced vector field $\tilde{X}_f = 0$ is completely integrable and thus the flow of X_f can be defined on $\mathbb{C} \times \mathcal{U}$ (c.f. [GW13] Lemma 5.2). The flow is given by $(\varphi_t)^*(z) = z + t \cdot f(z)\theta_1\theta_2$ and $(\varphi_t)^*(\theta_j) = \theta_j$. For any holomorphic functions f and g we have $[X_f, X_g] = 0$, and thus their flows commute (c.f. [Ber14], Corollary 3.8). Therefore, $\{X_f | f \in \mathcal{O}(U)\} \cong \mathcal{O}(U)$ is an uncountable infinite-dimensional abelian Lie algebra. If the set of automorphisms of $\mathcal{U}$ carried the structure of a Lie group such that its action on $\mathcal{U}$ was smooth, its Lie algebra would necessarily contain $\{X_f | f \in \mathcal{O}(U)\} \cong \mathcal{O}(U)$ as a Lie subalgebra, which is not possible.

9. Examples

In this section, we determine the automorphism group $\mathrm{Aut}(\mathcal{M})$ for some complex superman-ifolds $\mathcal{M}$ with underlying manifold $M = \mathbb{P}_1\mathbb{C}$.

Let L_1 denote the hyperplane bundle on $M = \mathbb{P}_1\mathbb{C}$ with sheaf of sections $\mathcal{O}(1)$, and $L_k = (L_1)^{\otimes k}$ the line bundle of degree k, $k \in \mathbb{Z}$, on $\mathbb{P}_1\mathbb{C}$, and sheaf of sections $\mathcal{O}(k)$. Each holomorphic vector bundle on $\mathbb{P}_1\mathbb{C}$ is isomorphic to a direct sum of line bundles $L_{k_1} \oplus \ldots \oplus L_{k_n}$ (see [Gro57]). Therefore, if $\mathcal{M}$ is a split supermanifold with $M = \mathbb{P}_1\mathbb{C}$ and $\dim \mathcal{M} = (1|n)$, there exist $k_1, \ldots, k_n \in \mathbb{Z}$ such that the structure sheaf $\mathcal{O}_\mathcal{M}$ of $\mathcal{M}$ is isomorphic to

$$\bigwedge(\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_n)).$$

Let $U_j = \{[z_0 : z_1] \in \mathbb{P}_1\mathbb{C} \,|\, z_j \neq 0\}$, $j = 1, 2$, and $\mathcal{U}_j = (U_j, \mathcal{O}_\mathcal{M}|_{U_j})$. Moreover, define $U_0^* = U_0 \setminus \{[1 : 0]\}$ and $U_1^* = U_1 \setminus \{[0 : 1]\}$, and let $\mathcal{U}_j^* = (U_j^*, \mathcal{O}_\mathcal{M}|_{U_j^*})$. We can now choose local coordinates $z, \theta_1, \ldots, \theta_n$ for $\mathcal{M}$ on U_0, and local coordinates $w, \eta_1, \ldots, \eta_n$ on U_1 so that the transition map $\chi : \mathcal{U}_0^* \to \mathcal{U}_1^*$, which determines the supermanifold structure of $\mathcal{M}$, is given by

$$\chi^*(w) = \frac{1}{z} \quad \text{and} \quad \chi^*(\eta_j) = z^{k_j}\theta_j.$$

Example 9.1. Let $\mathcal{M} = (\mathbb{P}_1\mathbb{C}, \mathcal{O}_\mathcal{M})$ be a complex supermanifold with $\dim \mathcal{M} = (1|1)$. Since the odd dimension is 1, the supermanifold $\mathcal{M}$ has to be split. Let $-k \in \mathbb{Z}$ be the degree of the associated line bundle. Choose local coordinates z, θ for $\mathcal{M}$ on U_0 and w, η on U_1 as above so that the transition map $\chi : \mathcal{U}_0^* \to \mathcal{U}_1^*$ is given by $\chi^*(w) = \frac{1}{z}$ and $\chi^*(\eta) = \frac{1}{z^k}\theta$.

We first want to determine the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of super vector fields on $\mathcal{M}$. A cal-culation in local coordinates verifying the compatibility condition with the transition map χ yields that the restriction to U_0 of any super vector field on $\mathcal{M}$ is of the form

$$\left((\alpha_0 + \alpha_1 z + \alpha_2 z^2)\frac{\partial}{\partial z} + (\beta + k\alpha_2 z)\theta\frac{\partial}{\partial \theta}\right) + \left(p(z)\frac{\partial}{\partial \theta} + q(z)\theta\frac{\partial}{\partial z}\right),$$

where $\alpha_0, \alpha_1, \alpha_2, \beta \in \mathbb{C}$, p is a polynomial of degree at most k, and q is a polynomial of degree at most $2 - k$. If $k < 0$ (respectively $2 - k < 0$), the polynomial p (respectively q) is 0. The Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even super vector fields is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, where an isomorphism $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ is given by

$$\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, d\right) \mapsto (-b - 2az + cz^2)\frac{\partial}{\partial z} + ((d - ka) + kcz)\theta\frac{\partial}{\partial \theta}.$$

Note that since the odd dimension of $\mathcal{M}$ is 1 each automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ gives rise to an automorphism of the line bundle L_{-k} and vice versa. Hence, the automorphism group $\mathrm{Aut}(L_{-k})$ of the line bundle L_{-k} and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ coincide.

A calculation yields that the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms $\mathcal{M} \to \mathcal{M}$ can be identified with $\mathrm{PSL}_2(\mathbb{C}) \times \mathbb{C}^*$ if k is even and with $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*$ if k is odd. Consider the element $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), s \right)$, where $s \in \mathbb{C}^*$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ is either an element of $\mathrm{SL}_2(\mathbb{C})$ or the representative of the corresponding class in $\mathrm{PSL}_2(\mathbb{C})$. The action of the corresponding element $\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is then given by

$$\varphi^*(z) = \frac{c + dz}{a + bz} \quad \text{and} \quad \varphi^*(\theta) = \left(\frac{1}{(a + bz)^k} + s \right) \theta$$

as a morphism over appropriate subsets of U_0 and by

$$\varphi^*(w) = \frac{aw + b}{cw + d} \quad \text{and} \quad \varphi^*(\eta) = \left(\frac{1}{(cw + d)^k} + s \right) \eta$$

over appropriate subsets of U_1.

The Lie supergroup structure on $\mathrm{Aut}(\mathcal{M})$ is now uniquely determined by $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $\mathrm{Vec}(\mathcal{M})$, and the adjoint action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$. Since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is connected, it is enough to calculate the adjoint action of $\mathrm{Vec}_{\bar{0}}(\mathcal{M}) \cong \mathfrak{sl}_2\mathbb{C} \oplus \mathbb{C}$ on $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$.

Let P_l denote the space of polynomials of degree at most l, and set $P_l = \{0\}$ for $l < 0$. The space of odd super vector fields $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$ is isomorphic to $P_k \oplus P_{2-k}$ via $\left(p(z) \frac{\partial}{\partial \theta} + q(z) \theta \frac{\partial}{\partial z} \right) \mapsto (p(z), q(z))$.

The element $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in \mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} \cong \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ corresponds to $-2z \frac{\partial}{\partial z} - k\theta \frac{\partial}{\partial \theta}$. The adjoint action of this super vector field on the first factor P_k of $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$ is given by by $-2z \frac{\partial}{\partial z} + k \cdot \mathrm{Id}$, and on the second factor P_{2-k} by $-2z \frac{\partial}{\partial z} + (2 - k) \cdot \mathrm{Id}$. Calculating the weights of the $\mathfrak{sl}_2(\mathbb{C})$-representation on P_k and P_{2-k}, we get that P_k is the unique irreducible $(k + 1)$-dimensional representation and P_{2-k} the unique irreducible $(3 - k)$-dimensional representation. Moreover, a calculation yields that $d \in \mathbb{C}$ corresponding to $d \cdot \theta \frac{\partial}{\partial \theta} \in \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ acts on P_k by multiplication with $-d$ and on P_{2-k} by multiplication with d.

If $k < 0$ or $k > 2$, we have

$$[\mathrm{Vec}_{\bar{1}}(\mathcal{M}), \mathrm{Vec}_{\bar{1}}(\mathcal{M})] = 0.$$

In the case $k = 0$, we have $P_k \cong \mathbb{C}$. Since $[\frac{\partial}{\partial \theta}, q(z) \theta \frac{\partial}{\partial z}] = q(z) \frac{\partial}{\partial z}$ for any $q \in P_2$, we get

$$[\mathrm{Vec}_{\bar{1}}(\mathcal{M}), \mathrm{Vec}_{\bar{1}}(\mathcal{M})] = \left\{ a(z) \frac{\partial}{\partial z} \ \middle| \ a \in P_2 \right\} \cong \mathfrak{sl}_2(\mathbb{C}),$$

and the map $P_0 \times P_2 \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(X, Y) \mapsto [X, Y]$, corresponds to $\mathbb{C} \times P_2 \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(p, q(z)) \mapsto p \cdot q(z) \frac{\partial}{\partial z}$.

Similarly, if $k = 2$, we have $P_{2-k} \cong \mathbb{C}$, and

$$[\mathrm{Vec}_{\bar{1}}(\mathcal{M}), \mathrm{Vec}_{\bar{1}}(\mathcal{M})] = \left\{ (\alpha_0 + \alpha_1 z + \alpha_2 z^2) \frac{\partial}{\partial z} + (\alpha_1 + 2\alpha_2 z) \theta \frac{\partial}{\partial \theta} \ \middle| \ \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C} \right\} \cong \mathfrak{sl}_2(\mathbb{C})$$

since $[p(z) \frac{\partial}{\partial \theta}, \theta \frac{\partial}{\partial z}] = p(z) \frac{\partial}{\partial z} + p'(z) \theta \frac{\partial}{\partial \theta}$, and the map $P_2 \times P_0 \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(X, Y) \mapsto [X, Y]$, corresponds to $P_2 \times \mathbb{C} \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(p(z), q) \mapsto q \cdot p(z) \frac{\partial}{\partial z} + q \cdot p'(z) \theta \frac{\partial}{\partial \theta}$.

If $k = 1$, then $P_k \oplus P_{2-k} \cong \mathbb{C}^2 \oplus \mathbb{C}^2$. We have

$$\left[\tfrac{\partial}{\partial \theta}, \theta \tfrac{\partial}{\partial z} \right] \quad = \quad \tfrac{\partial}{\partial z}, \qquad \left[z \tfrac{\partial}{\partial \theta}, \theta \tfrac{\partial}{\partial z} \right] \quad = \quad z \tfrac{\partial}{\partial z} + \theta \tfrac{\partial}{\partial \theta},$$

$$\left[\tfrac{\partial}{\partial \theta}, z \theta \tfrac{\partial}{\partial z} \right] \quad = \quad z \tfrac{\partial}{\partial z}, \qquad \left[z \tfrac{\partial}{\partial \theta}, z \theta \tfrac{\partial}{\partial z} \right] \quad = \quad z^2 \tfrac{\partial}{\partial z} + z \theta \tfrac{\partial}{\partial \theta},$$

and consequently $[\mathrm{Vec}_{\bar{1}}(\mathcal{M}), \mathrm{Vec}_{\bar{1}}(\mathcal{M})] = \mathrm{Vec}_{\bar{0}}(\mathcal{M})$.

Remark that $\mathrm{Aut}(\mathcal{M})$ carries the structure of a split Lie supergroup if and only if $k < 0$ or $k > 2$ (c.f. Proposition 4 in [Vis11]).

Example 9.2. Let $\mathcal{M} = (\mathbb{P}_1\mathbb{C}, \mathcal{O}_\mathcal{M})$ be a split complex supermanifold of dimension $\dim \mathcal{M} = (1|2)$ associated to $\mathcal{O}(-k_1) \oplus \mathcal{O}(-k_2)$, $k_1, k_2 \in \mathbb{Z}$. We will determine the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms $\mathcal{M} \to \mathcal{M}$.

We choose coordinates z, θ_1, θ_2 for $\mathcal{U}_0$ and w, η_1, η_2 for $\mathcal{U}_1$ as described above such that the transition map χ is given by $\chi^*(w) = z^{-1}$ and $\chi^*(\eta_j) = z^{-k_j}\theta_j$.

The action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{P}_1\mathbb{C}$ by Möbius transformations lifts to an action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{M}$ by letting $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ act by the automorphism $\varphi_A : \mathcal{M} \to \mathcal{M}$ with pullback

$$\varphi_A^*(z) = \frac{c + dz}{a + bz} \quad \text{and} \quad \varphi_A^*(\theta_j) = (a + bz)^{-k_j}\theta_j$$

as a morphism over appropriate subsets of U_0, and

$$\varphi_A^*(w) = \frac{aw + b}{cw + d} \quad \text{and} \quad \varphi_A^*(\eta_j) = (cw + d)^{-k_j}\eta_j$$

over appropriate subsets of U_1. Using the transition map χ one might also calculate the representation of φ in coordinates as a morphism over subsets $U_0 \to U_1$ and $U_1 \to U_0$.

If k_1 and k_2 are both even, we have $\varphi_A = \mathrm{Id}_\mathcal{M}$ for $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and thus we get an action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{M}$.

Consider the homomorphism of Lie groups $\Psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(\mathbb{P}_1\mathbb{C})$ assigning to each automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ the underlying biholomorphic map $\tilde{\varphi} : \mathbb{P}_1\mathbb{C} \to \mathbb{P}_1\mathbb{C}$. This homomorphism Ψ is surjective since $\mathrm{Aut}(\mathbb{P}_1\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$ and since the $\mathrm{PSL}_2(\mathbb{C})$-action on $\mathbb{P}_1\mathbb{C}$ lifts to an action (of $\mathrm{SL}_2(\mathbb{C})$) on the supermanifold $\mathcal{M}$. The kernel $\ker \Psi$ of the homomorphism Ψ consists of those automorphisms $\varphi : \mathcal{M} \to \mathcal{M}$ whose underlying map $\tilde{\varphi}$ is the identity $\mathbb{P}_1\mathbb{C} \to \mathbb{P}_1\mathbb{C}$. This kernel $\ker \Psi$ is a normal subgroup, $\mathrm{SL}_2(\mathbb{C})$ acts on $\ker \Psi$, and we have

$$\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \Psi \rtimes \mathrm{SL}_2(\mathbb{C})$$

if k_1 and k_2 are not both even, and $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C})$ if k_1 and k_2 are even. Thus, it remains to determine $\ker \Psi$.

Let $\varphi : \mathcal{M} \to \mathcal{M}$ be an automorphism with $\tilde{\varphi} = \mathrm{Id}$. Let f and b_{jk}, $j, k = 1, 2$, be holomorphic functions on $U_0 \cong \mathbb{C}$ such that the pullback of φ over U_0 is given by

$$\varphi^*(z) = z + f(z)\theta_1\theta_2 \quad \text{and} \quad \varphi^*(\theta) = B(z)\theta,$$

where $B(z) = \left(\begin{smallmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{smallmatrix}\right)$ and $\varphi^*(\theta) = B(z)\theta$ is an abbreviation for $\varphi^*(\theta_j) = b_{j1}(z)\theta_1 + b_{j2}(z)\theta_2$ for $j = 1, 2$. Similarly, let g and c_{jk} be holomorphic functions on $U_1 \cong \mathbb{C}$ such that the pullback of φ over U_1 is given by

$$\varphi^*(w) = w + g(w)\eta_1\eta_2 \quad \text{and} \quad \varphi^*(\eta) = C(z)\eta,$$

where $C(z) = \left(\begin{smallmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{smallmatrix}\right)$. The compatibility condition with the transition map χ gives now the relation

$$f(z) = -z^{2-(k_1+k_2)}g\left(\frac{1}{z}\right) \quad \text{for } z \in \mathbb{C}^*.$$

Therefore, f and g are both polynomials of degree at most $2 - (k_1 + k_2)$, and they are 0 in the case $k_1 + k_2 > 2$. For the matrices B and C we get the relation

$$B(z) = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} C\left(\frac{1}{z}\right) \begin{pmatrix} z^{-k_1} & 0 \\ 0 & z^{-k_2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}^*.$$

If $k_1 = k_2$, this implies $B(z) = C\left(\frac{1}{z}\right)$ for all $z \in \mathbb{C}^*$. Thus, $B(z) = B$ and $C(w) = C$ are constant matrices, and $B = C \in \mathrm{GL}_2(\mathbb{C})$ since φ was assumed to be invertible. Consequently, we have

$$\ker \Psi \cong P_{2-(k_1+k_2)} \rtimes \mathrm{GL}_2(\mathbb{C})$$

in the case $k_1 = k_2$, where $P_{2-(k_1+k_2)}$ denotes the space of polynomials of degree at most $2 - (k_1 + k_2)$ if $k_1 + k_2 < 2$ and $P_{2-(k_1+k_2)} = \{0\}$ otherwise. The group structure on the semidirect product is given by $(f_1(z), B_1) \cdot (f_2(z), B_2) = (\det B_1 f_1(z) + f_2(z), B_1 B_2)$.

Let now $k_1 \neq k_2$. After possibly changing coordinates we may assume $k_1 > k_2$. Then we have

$$B(z) = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} C\left(\frac{1}{z}\right) \begin{pmatrix} z^{-k_1} & 0 \\ 0 & z^{-k_2} \end{pmatrix} = \begin{pmatrix} c_{11}\left(\frac{1}{z}\right) & z^{k_1-k_2} c_{12}\left(\frac{1}{z}\right) \\ z^{k_2-k_1} c_{21}\left(\frac{1}{z}\right) & c_{22}\left(\frac{1}{z}\right) \end{pmatrix}$$

for all $z \in \mathbb{C}^*$. This implies that $b_{11} = c_{11}$ and $b_{22} = c_{22}$ are constants. Since we assume $k_1 > k_2$, we also get $b_{21} = c_{21} = 0$ and b_{12} and c_{12} are polynomials of degree at most $k_1 - k_2$. Therefore,

$$\ker \Psi \cong P_{2-(k_1+k_2)} \rtimes \left\{ \begin{pmatrix} \lambda & p(z) \\ 0 & \mu \end{pmatrix} \middle| \lambda, \mu \in \mathbb{C}^*, p \in P_{k_1-k_2} \right\},$$

and the group structure is again given by $(f_1(z), B_1) \cdot (f_2(z), B_2) = (\det B_1 f_1(z) + f_2(z), B_1 B_2)$ for $f_1, f_2 \in P_{2-(k_1+k_2)}$, $B_1, B_2 \in \left\{ \begin{pmatrix} \lambda & p(z) \\ 0 & \mu \end{pmatrix} \middle| \lambda, \mu \in \mathbb{C}^*, p \in P_{k_1-k_2} \right\}$.

The semidirect product $\ker \Psi \rtimes \mathrm{SL}_2(\mathbb{C})$ (or $\ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C})$) is a direct product if and only if $k_1 = k_2$ and $k_1 + k_2 \geq 2$.

Example 9.3. Let $\mathcal{M} = (\mathbb{P}_1\mathbb{C}, \mathcal{O}_{\mathcal{M}})$ be the complex supermanifold of dimension $\dim \mathcal{M} = (1|2)$ given by the transition map $\chi : \mathcal{U}_0^* \to \mathcal{U}_1^*$ with pullback

$$\chi^*(w) = \frac{1}{z} + \frac{1}{z^3} \theta_1 \theta_2 \quad \text{and} \quad \chi^*(\eta_j) = \frac{1}{z^2} \theta_j.$$

The supermanifold $\mathcal{M}$ is not split and the associated split supermanifold corresponds to $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$; see e.g. [BO96].

As in the previous example, the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{P}_1\mathbb{C}$ by Möbius transformations lifts to an action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{M}$. Let A denote the class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{PSL}_2(\mathbb{C})$. Then A acts by the morphism $\varphi_A : \mathcal{M} \to \mathcal{M}$ whose pullback as a morphism over appropriate subsets of U_0 is given by

$$\varphi_A^*(z) = \frac{c + dz}{a + bz} - \frac{b}{(a + bz)^3} \theta_1 \theta_2 \quad \text{and} \quad \varphi_A^*(\theta_j) = \frac{1}{(a + bz)^2} \theta_j.$$

Let $\Psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(\mathbb{P}_1\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$ denote again the Lie group homomorphism which assigns to an automorphism of $\mathcal{M}$ the underlying automorphism of $\mathbb{P}_1\mathbb{C}$. The assignment $A \mapsto \varphi_A \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ defines a section $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of Ψ, and we have

$$\mathrm{Aut}_{\bar{0}}(\mathbb{C}) \cong \ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C}).$$

The section $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ induces on the level of Lie algebras the morphism $\sigma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, which maps an element $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$ to the super vector field on $\mathcal{M}$ whose restriction to $\mathcal{U}_0$ is

$$\left(c - 2az - bz^2 - b\theta_1\theta_2\right) \frac{\partial}{\partial z} - 2(a + bz) \left(\theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2}\right).$$

We now calculate the kernel $\ker \Psi$. Let $\varphi \in \ker \Psi$. Its underlying map $\tilde{\varphi}$ is the identity and we thus have

$$\varphi^*(z) = z + a_0(z)\theta_1\theta_2 \quad \text{and} \quad \varphi^*(\theta) = A_0(z)\theta$$

on U_0 and

$$\varphi^*(w) = w + a_1(w)\eta_1\eta_2 \quad \text{and} \quad \varphi^*(\eta) = A_1(w)\eta$$

on U_1 for holomorphic functions a_0 and a_1 and invertible matrices A_0 and A_1 whose entries are holomorphic functions. The notation $\varphi^*(\theta) = A_0(z)\theta$ (and similarly $\varphi^*(\eta) = A_1(w)\eta$) is again an abbreviation for $\varphi^*(\theta_j) = (A_0(z))_{j1}\theta_1 + (A_0(z))_{j2}\theta_2$, where $A_0(z) = ((A_0(z))_{jk})_{1 \leq j,k \leq 2}$. A calculation with the transition map χ then yields the relations

$$A_1(w) = A_0\left(\frac{1}{w}\right) \quad \text{and} \quad a_1(w) = \frac{1}{w}\left(\left(\det A_0\left(\frac{1}{w}\right) - 1\right) - \frac{1}{w}a_0\left(\frac{1}{w}\right)\right)$$

for any $w \in \mathbb{C}^*$. Since a_0, a_1, A_0, and A_1 are holomorphic on $\mathbb{C}$, we get that $A_0 = A_1$ are constant matrices, $\det A_0 = 1$, and $a_0 = a_1 = 0$. Therefore, $\ker \Psi \cong \mathrm{SL}_2(\mathbb{C})$, and its Lie algebra is

$$\mathrm{Lie}(\ker \Psi) = \left\{ (a_{11}\theta_1 + a_{12}\theta_2)\frac{\partial}{\partial\theta_1} + (a_{21}\theta_1 + a_{22}\theta_2)\frac{\partial}{\partial\theta_2} \,\middle|\, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \right\}.$$

Since $\mathrm{Lie}(\ker \Psi)$ and $\sigma(\mathrm{Lie}(\mathrm{PSL}_2(\mathbb{C}))$ commute, the semidirect product $\ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C})$ is direct and we have

$$\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C}).$$

Remark in particular that this group is different from the automorphism group of the corresponding split supermanifold $\mathcal{N}$, which is associated to $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, with $\mathrm{Aut}_{\bar{0}}(\mathcal{N}) \cong \mathrm{GL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$.

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Fakultät für Mathematik, Ruhr-Universität Bochum, Universitätsstr. 150, D-44780 Bochum, Germany
E-mail address: `Hannah.Bergner-C9q@rub.de`
E-mail address: `Matthias.Kalus@rub.de`